The Commutative Cochain Problem

A post from the blog Just Categories

BY J. SANCHEZ

Associated to a topological space \( X \) there is an algebraic structure, the cohomology algebra \( H^*(X; k) \). The structure \( H^*(X; k) \) can be seen as a simplified algebraic version of \( X \), which focuses on topological information of the space preserved by homotopy equivalences.

Because of its huge amount of applications, the computation of this structures has become an important problem in mathematics. But, there are some inconveniences, in general this is not a simple structure and the algebra's behavior may change with the ring \( k \). Also, the analysis techniques vary with the characteristic of the ring.

To study the cohomology ring of a space \( X \) it is useful to look at the previous stage of the construction, that is, understand the behavior of the normalized cochain ring \( C^*(X; k) \). In this part, the operation used as multiplication, the cup product of cochains, usually is not commutative. In the graded case, commutativity means that \( x \cup y = (-1)^{|x||y|} y \cup x \), where \(|x|\) and \(|y|\) are the degrees of \( x \) and \( y \). So, we can say that the general picture of \( C^*(X; k) \) is a graded set of cochains over \( k \), where the elements of degree \( n \) form \( k \)-modules \( C^n(X; k) \) connected each other by a differential \( \partial \), and with a graded multiplication or product.

Much of the information in \( C^*(X; k) \) will be lost when we take the homology to create \( H^*(X; k) \). So, it is quite plausible to ask for a simpler structure to replace \( C^*(X; k) \). This new structure need to fill some natural requirements, like: we have to obtain the same ring \( H^*(X; k) \) but isomorphism when we calculate its homology. And one desirable property is that it must have a commutative product.

Essentially, the commutative cochain problem is to find functorially a differential graded algebra \( A^*(X) \) over the ring \( k \), such that there is a quasi isomorphism between \( A^*(X) \) and \( C^*(X; k) \). There are requirements over \( A^*(X) \) which are not filled by \( H^*(X; k) \), as when \( Y \) is a subspace of \( X \), we must have an epimorphism from \( A^*(Y) \) to \( A^*(X) \).

For the study of this problem we could use a technical approach to topological spaces, and work directly over the category of simplicial sets. This means that we replace topological spaces for simplicial sets, and more precisely punctual finite simplicial sets. This kind of simplicial sets are used to represent simply connected
topological spaces. The change of category is possible because of the Dold-Kan correspondence, which stated that the category of chain complexes is equivalent to the category of simplicial abelian groups.

The commutative cochain problem was solved for the rational case by Daniel Quillen and later, in a second approach by Dennis Sullivan. The method used by Sullivan introduced the notion of minimal model associated to a graded commutative differential algebra. The importance of his method is that the minimal model is an homotopy invariant that characterize the rational homotopy type of the space.

The models proposed by Sullivan simplified enormously the computations in rational homology. Also his works helped to expand the branch in mathematics known as Rational Homotopy Theory. For the curious reader, Katherine Hess wrote a overview document about the subject, titled Rational Homotopy Theory: A Brief Introduction.

The algebra used to model the rational cohomology of an space is called polynomial differential forms with coefficients in $k$, noted $A_{PL}$. It is a simplicial commutative cochain algebra. $A_{PL}$ assign to each simplicial set $X$ a commutative cochain algebra $A_{PL}(X)$, known as the cochain algebra of polynomial differential forms on $X$ with coefficients in the field $k$.

The quasi isomorphism of cochain complexes between $A_{PL}(X)$ and $C^*(X; k)$ is a kind of integration. Constructed for smooth manifolds and differential forms in the 1930’s, this quasi isomorphism have the particularity that it commutes differentials. That property is the well know Stoke’s theorem.

As one might expect, in the $\mathbb{R}$ case, when $M$ is a smooth manifold, there exist a quasi isomorphism between $A_{PL}(M)$ and the classical cochain algebra $A_{DL}(M)$ of smooth differential forms. Recall that $A_{DL}(M)$ is used to construct the de Rham cohomology of the smooth manifold $M$.

In 1947, the American mathematician Norman Steenrod proposed a new type of cohomology operations that generalize the cup product. The Steenrod squares $Sq^i$ are defined over the cohomology ring with coefficient in $\mathbb{Z}/2\mathbb{Z}$. They take the class $x$ of a cocycle of degree $n$ in to a class $Sq^i(x)$ of degree $n + i$. When $n = i$, $Sq^i(x)$ is just the cup product $x \cup x$.

The construction of the Steenrod squares depends strongly on the non commutativity of the cochain ring $C^*(X; \mathbb{Z})$. The key observation is that even if $x \cup y$ is not equal to $(-1)^{|x||y|} y \cup x$, $\cup$ is commutative up to homotopy. This give a new kind of cup product, noted $\cup_1$, which works as the homotopy between the non commutative terms. Again, this 1-cup product $\cup_1$ is not commutative but commutative up to homotopy, with an homotopy denoted $\cup_2$. Repeating again and again the process we obtain a collection of cup products, $\cup_i$, with $i \geq 0$. 

"tori"
Then with the $i$-cup products we define the operations called $i$-squares, $Sq_i(x) = x \cup_i x$, with the under index. It is important to notice that the operations $Sq_i$ are defined in cochains over the integers. Now, as a consequence that the coboundary of $Sq_i(x)$, where $x$ is a module 2 cocycle, is again a module 2 cocyle, we are able to take the operations $Sq_i$ to $H^*(X; \mathbb{Z}/2\mathbb{Z})$. And finally the Steenrod squares appear as a rearrangement of indexes, $Sq^i := Sq_{*-i}$.

One consequence of the existence of the Steenrod operations is that there is no solution for the commutative cochain problem over the integer. We can see this implication in the next informal way. Suppose that you have a differential commutative cochain algebra $A^*(X)$ quasi isomorphic to $C^*(X; \mathbb{Z})$. In $C^*(X; \mathbb{Z})$ we have the $i$-cup products and these operations can be carried to $A^*(X)$ by the quasi isomorphism and they homotopically coincide with the respective $i$-cup products in $A^*(X)$. But there, in $A^*(X)$, the product is commutative, so the $i$-cup products are zero, and when we take the cohomology over $\mathbb{Z}/2\mathbb{Z}$, we obtain operations which are zero in $H^*(X; \mathbb{Z}/2\mathbb{Z})$. But from the Steenrod construction they are not zero. So, if we assume the existence of a solution for the problem we obtain that $Sq^i = 0$ in the $\mathbb{Z}/2\mathbb{Z}$ cohomology ring of $X$.

Clearly, in order to be convinced, one may need an example of space $X$, in which $Sq^i(x) \in H^*(X; \mathbb{Z}/2\mathbb{Z})$ is not zero for some cocycle class $x$ in $C^*(X; \mathbb{Z})$, using the definition of Steenrod. Bohumil Cenkl in his article *Cohomology Operations From Higher Products In The DeRham Complex*, make the example using the circle $S^1$ and the cocycle in $C^1(S^1; \mathbb{Z})$ with cohomology class the identity homomorphism in $\mathbb{Z}$.

And he obtains the contradiction that 1 is divisible by 2 in $\mathbb{Z}$.

Many articles have been written with the intention to generalize the Sulliva’s ideas to cohomologies with arbitrary coefficients. For example, the French mathematician Henry Cartan in his article *Théories Cohomologiques* describes a possible axiomatization for a simplicial differential algebra $A^*$ in order to obtain a natural isomorphism between the cohomology of $A^*(X)$, where $X$ is a simplicial set, and $H^*(X; k)$. The ring $k$ depends on the choice of $A^*$.

Contemporary approaches use the theory of operads. They look for a more detailed description of algebras by the use of $E_\infty$ algebras. This is due at the complexity in the case of fields of positive characteristic.

The commutative cochain problem opened a vast field of new ideas and concepts. And it confirms that if we want to know the behavior of a problem, the best approaches are the ones that use the connections between the different branches in mathematics.

For more information about the subject: