The concept of minimal model in rational homotopy theory was created by Dennis Sullivan in the 1960’s. He discovered in the category of topological spaces, that simply connected spaces can be rationalized. Which means we can replace an space $X$ with a rational version of it, $X\mathbb{Q}$, such that $H_*(X; \mathbb{Q}) = H_*(X\mathbb{Q})$.

Recall that a simply connected space $Y$ is say to be rational when its reduced homology (or $\pi_*(Y)$, or the reduced homology of its loop space $\Omega Y$) is a $\mathbb{Q}$-vector space. When we find for a simply connected space $X$ an rational space $Y$(which is again simply connected) and a continuous map $\phi : X \to Y$ that induces the homotopy morphism $\pi_*(\phi) \otimes \mathbb{Q} : \pi_*X \otimes \mathbb{Q} \to \pi_*Y \otimes \mathbb{Q} = \pi_*Y$, the simply connected space $Y$ is called a rationalization of $X$.

When we work with simply connected spaces, the existence of rationalizations is guaranteed. Moreover, given a continuous map $\phi : X \to Z$, we can state the existence of an (up to homotopy) unique induced morphism between the rationalizations of $X$ and $Z$. With this, the rational homotopy type of a simply connected space is defined as the weak homotopy of its rationalization. Then, $\phi : X \to Z$ is called a rational homotopy equivalence when the induced rational map is a weak homotopy equivalence.

This simplification of an space implies some lost of information about it, for example, the homotopy groups of the sphere $S^2$ are non-zero in infinitely many degrees, but the rational homotopy groups vanish in all degrees above 3. But, the advantage of the approximation by a rational model, is the facility for computations when ordinary homotopy theory is too complicated.

According to the book *Rational Homotopy Theory* by Yves Félix, Stephen Halperin and Jean-Claude Thomas, one important early result (by Vigué and Sullivan) using rational homotopy, says that if $M$ is a simply connected compact riemannian manifold whose rational cohomology algebra requires at least two generators, then its free loop space has unbounded homology and hence $M$ has infinitely many geometrically distinct closed geodesics.

But, where comes from the easier calculations? Well, this is due to Quillen and Sullivan, they discovered an explicit algebraic formulation for rational homotopy. Which means that the rational homotopy type of a topological space is the same as the isomorphism class of its algebraic model. Moreover, in each class there exists a well defined representative, which is called minimal model. This object is indeed a special kind of differential graded algebra.

Let $X$ a simply connected rational space. Now, we are interesting in describe things like its cohomology $H^*(X)$. But
first, we need to compute $C^*(X)$, which is in general a structure with a non commutative product. This detail increase the difficult for compute cohomology, but Sullivan found a functor $A_{PL}$ that associate a commutative cochain algebra $A_{PL}(X)$ to $X$. The link between both structures is expressed by two quasi-isomorphisms, $C^*(X) \rightarrow D(X)$ and $A_{PL}(X)$. So we have that $H^*(X) = H(A_{PL}(X))$.

The functor $A_{PL}$ is constructed following the next steps. First consider a collection of cochain algebras $\{A_n\}_{n \geq 0}$ that forms, with plausible face and degeneracy maps, a simplicial object $A$ in the category of cochain algebras. For a given integer $p$, the elements of order $p$ in each cochain algebra $A_n$, $n \geq 0$, together with the restriction of the face and degeneracy maps, forms a simplicial set denoted $A^p$. Given a simplicial set $K$, and a simplicial cochain algebra $A$, we can form a simplicial cochain algebra $A(K)$, which set of elements of order $p$, are the sets of simplicial maps from $K$ to $A^p$.

After that, the $A_{PL}$ is obtained from a special choice of $A$(using the notation in the last paragraph), and $A_{PL}(X) = A_{PL}(S_*(X))$. Here $S_*(X)$ is the simplicial set of singular simplices associated to the topological space $X$. For $A_{PL}$, the cochain algebra $(A_{PL})_n$ is the given by the quotient of the free graded commutative algebra $\Lambda(t_0, \ldots, t_n, y_0, \ldots, y_n)$, where the basis elements $t_i$ have degree zero and the basis elements $y_i$ have degree 1, and the ideal generated by the two elements $\sum_0^n t_i - 1$ and $\sum_0^n y_j$. The commutative cochain algebra $A_{PL}(X)$ is called the cochain algebra of polynomial differential forms on $X$ with coefficients in $Q$. This construction was suggested by the classical cochain algebra $A_{DR}(M)$ of smooth differential forms on a smooth manifold $M$.

The transition from topological spaces to commutative cochain algebras established by the functor $A_{PL}$ allows to focus in the study of commutative cochain algebras themselves. In this category shows up a special kind of commutative cochain algebras, they are called Sullivan algebras. This algebras lives in each isomorphism class, and under special conditions over the space $X$, have a minimal representative.

A Sullivan algebra is a commutative cochain algebra of the form $(\Lambda V, d)$, where $V = \{V^p\}_{p \geq 1}$, that is a collection of sets of elements with different degrees, and $V$ can be expressed as an union $\bigcup_{k=0}^{\infty} V(k)$ with $V(0) \subset V(1) \subset \ldots$ an increasing sequence of graded subspaces. The differential satisfies $d = 0$ in $V(0)$ and $d : V(k) \rightarrow \Lambda V(k-1)$, for $k \geq 1$.

The last condition is called the nilpotent condition on $d$, and it can be restated saying that: $d$ preserves $\Lambda V(k)$ and there exist graded subspaces $V_k \subset V(k)$ with $\Lambda V(k) = \Lambda V(k-1) \otimes \Lambda V_k$, and $d : V_k \rightarrow \Lambda V(k-1)$.

With this, we can state now the principal definition of this article: a Sullivan model for a commutative cochain algebra $(A, d)$ is a quasi-isomorphism $m$ from a Sullivan algebra $(\Lambda V, d)$ to $(A, d)$. In the case of $A_{PL}(X)$, with $X$ a path connected topological space, a Sullivan model for the algebra $A_{PL}(X)$, is called a Sullivan model for $X$. Moreover, this
model is called \textit{minimal} is the differential satisfies $Im(d) \subset \Lambda^+V \cdot \Lambda^+V$. When the algebra $(A,d)$ is such that $H^0(A) = \mathbb{Q}$, there always exists a minimal Sullivan model, and this is uniquely determined up to isomorphism.

At this point we can see the possible simplification made by the minimal models: if $(\Lambda V,d)$ is a Sullivan minimal model for the rational space $X$ then $H(\Lambda V,d)$ is isomorphic to $H(A_{PL}(X))$ and isomorphic to $H^*(X)$.

It is important to say that if simply connected topological spaces $X$ and $Y$ have the same rational homotopy type, then the cochain algebras $A_{PL}(X)$ and $A_{PL}(Y)$ are weakly equivalent, and by the unicity of the minimal models, they have the same minimal model. So, if we restrict to simply connected spaces with rational homology of finite type, there is a bijection between the rational homotopy types with the isomorphism classes of minimal Sullivan algebras over $\mathbb{Q}$,($(\Lambda V,d)$ with $V^1 = 0$.

Now, let’s see some examples. With $k$ odd, the minimal Sullivan model for the sphere $S^k$ is $m : (\Lambda(e),0) \to A_{PL}(S^k)$, where the degree of $e$ is $k$ and the image of $e$ by $m$ is the representing cocycle of the class in $H^k(A_{PL}(S^k))$, determined by the fundamental class $[S^k] \in H_k(S^k;\mathbb{Z})$. For a product of spaces $X \times Y$, the minimal model is the tensor product of the minimal models of each space.

Recall that a based topological space $(X,\ast)$ with a continuous map $\mu : X \times X \to X$, with the maps $x \mapsto \mu(x,\ast)$ and $x \mapsto \mu(\ast,x)$ homotopic to the identity, is called an $H$-space. This kind of spaces have a minimal Sullivan models of the form $(\Lambda V,0)$, that is, with null differential.

Also, they are cochain algebras $(\Lambda V,d)$ which are not Sullivan algebras. For example, let $V = \{v_1,v_2,v_3\}$, with degree of $v_1$, 1, $dv_1 = v_2v_3$, $dv_2 = v_3v_1$ and $dv_3 = v_1v_2$. But, a cochain algebra $(\Lambda V,d)$, where $V = V^{\leq 2}$ and $Im(d) \subset \Lambda^+V \cdot \Lambda^+V$, is always a minimal Sullivan algebra.

Once the Sullivan model notion established, the research focuses in the study of models for geometric constructions using the models of the spaces participating in the construction as we can saw in the example of the product. Another interesting geometric construction to do this are, for example, cone attachments, cell attachments ans suspensions.

There is a generalization of Sullivan algebras, called relative Sullivan algebras. This generalization comes from the observation that cochain algebras are equipped with a morphism $Q \to (A,d)$ which is particular case of the morphism of commutative cochain algebras $(B,d) \to (A,d)$, so the notion of Sullivan algebra for $(A,d)$ is extended to the notion of relative Sullivan algebra for this morphism. And they have the form $(B \otimes \Lambda V,d)$, where $H^0(B)$ is equal to $\mathbb{Q}$ and there is a quasi-isomorphism of cochain algebras from $(B \otimes \Lambda V,d)$ to $(C,d)$.

The generalization is useful to model objects like fibrations. Let $F$ be the fiber of a continuous map $f : X \to Y$, where $Y$ is simple connected with rational homology of finite type, then it is possible to construct a Sullivan model for $F$ directly from the morphism $A_{PL}(f) : A_{PL}(Y) \to A_{PL}(X)$. And, of course, using this technique it is possible to construct the Sullivan model of new interesting spaces.

More information about the subject,


